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On using materially conserved quantities to construct solutions of differential equations

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Abstract. In this article we consider how one may *systematically* determine materially conserved quantities (MCQs) from the equations describing the dynamics of a fluid and then how these can be used to construct quite general, exact, analytical solutions of these equations. Such solutions necessarily reflect underlying physical processes and are often general enough to satisfy strong boundary conditions. The method described for determining MCQs is systematic and essentially algorithmic, and is therefore a good candidate for implementation using a computer-algebra system.

We use this method to recover three MCQs of the equations describing large-scale fluid flow on the surface of a rotating sphere (geostrophic flow). We then prove that these geostrophic equations admit no further MCQs. Next, we describe how to construct fairly general, exact solutions from these quantities.

Finally we discuss the application of these ideas to general systems of fluid-dynamical equations, in particular we consider under what conditions there exist MCQs which hold for a significant number of solutions of a system of governing equations and when these can be found without *a priori* determining these solutions.

1. Introduction

It is well known that the dynamics defined by the equations which describe large-scale flow of a thin layer of fluid on the surface of a rotating sphere (e.g. the world's oceans), the *geostrophic equations*, admit certain materially conserved quantities (MCQs). These quantities have been used in various *ad hoc* ways to construct solutions of the equations. A systematic approach is needed. Moreover, this approach should be applicable to other fluid systems. In this paper we discuss how one can *systematically* determine the MCQs of a wide class of fluid-dynamical systems and how they may be used to construct surprisingly general analytical solutions.

Given n MCQs of a system of partial differential equations (PDEs) in n independent variables the MCQs are necessarily functionally related. These relations take the form of differential equations which are significantly simpler than the governing equations. Solution of these equations yields classes of solution ansatz which necessarily reflect fundamental underlying physics and which satisfy strong boundary conditions—stronger than those which may be satisfied by solutions obtained from point symmetries (e.g. Bluman and Kumei 1989, Olver 1986, Stephani 1990); see section 5 for further discussion on this.

We show how one can search, in a systematic, essentially algorithmic way, for such MCQs directly from the governing equations of a large class of fluid-dynamical systems.

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The method is a good candidate for implementation using a computer-algebra system such as MAPLE or MACSYMA. We apply this method to the geostrophic equations. We then discuss the number of MCQs a given system can have. We show that the geostrophic equations have only three MCQs (to the author's knowledge no such proof has been previously given) and that in general a system of equations in n independent variables has n MCQs, though it is not obvious, in general, how to find these systematically without first solving the governing equations; we consider under what circumstances it is possible to do this.

The class of solutions which one finds from the MCQs of a system depends on the function relating them; this function is arbitrary. Hence the question of which such functions lead to physical solutions thus arises. We finally describe how one can do this by introducing additional physical processes into the system.

2. Background

The equations most often used to describe large-scale motion of a thin layer of fluid on the surface of a rotating sphere are, in nondimensional form,

$$u = -\frac{p_y}{y} \qquad v = \frac{p_x}{y} \qquad \rho = p_z \tag{2.1a}$$

$$u_x + v_y + w_z = 0 \tag{2.1b}$$

and

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{2.1c}$$

(three momentum equations, continuity and a thermodynamic equation, respectively), where u, v and w are the components of the velocity field in the x, y and z directions, ρ is density, and p is pressure. They may be written

$$M_x M_{zzz} + y(M_{xz}M_{yzz} - M_{yz}M_{xzz}) = 0 \tag{2.2a}$$

$$u = -\frac{M_{yz}}{y} \qquad v = \frac{M_{xz}}{y} \qquad w = \frac{M_x}{y^2} \qquad p = M_z \qquad \rho = M_{zz} \tag{2.2b}$$

(Welander 1959). These equations are well known to admit the three MCQs

$$\rho = M_{zz} \qquad B = M_z - zM_{zz} \qquad \text{and} \qquad q = yM_{zzz}. \tag{2.3}$$

Physically, this means that these three quantities take a constant value on the trajectory of every fluid parcel. Each may be interpreted physically: $\rho = M_{zz}$ is the density of the fluid; $B = M_z - zM_{zz}$ is the Bernoulli function which is associated with the energy per unit mass of the fluid (e.g. Batchelor 1967); $q = yM_{zzz}$ is the potential vorticity, a quantity related to the usual vorticity of a fluid (e.g. Pedlosky 1987).

Welander (1971) showed that these three MCQs must be functionally related, i.e. that

$$yM_{zzz} = \mathcal{M}(M_{zz}, M_z - zM_{zz}) \tag{2.4}$$

for some function $\mathcal{M}$. Significantly, (2.4) is an *ordinary* differential equation (ODE) for $M(z; x, y)$ (x and y play only a parametric role). Hence, if this function can be determined then we have only to integrate an ODE and ensure consistency with the governing equation, (2.2a). This is a huge simplification of the problem.

Motivated by these ideas, we would like some systematic method of determining MCQs of a system of governing equations and also a method of relating these which, ideally, reflects the physics of the situation and leads to classes of solutions which are relevant. Furthermore, we would like these methods to have some degree of generality, i.e. be applicable to other systems of PDEs. We address these issues in the following sections.

Notation. The following notation is used: $\partial_{\boldsymbol{x}} u = u_x, u_y, u_z$ and $M_{\boldsymbol{x}} = M_x, M_y, M_z$, i.e. boldface subscripts indicate all derivatives of the corresponding order. Hence $M_{\boldsymbol{xx}} = M_{xx}, M_{xy}, \ldots$ etc.

3. Determining MCQs

In this section we first introduce formally the concept of material conservation laws of a differential equation. We then show how one can directly search for such laws and recover the three quantities, (2.3). Finally we prove that no more MSQs of the geostrophic equations, (2.2), exist.

Motivated by the conservation of physical quantities such as energy and momentum, a dynamical quantity $Q = \mathcal{Q}(t, \boldsymbol{s}(t), \dot{\boldsymbol{s}}(t))$ (where $\boldsymbol{s} = (s_1, s_2, \ldots)$ are the trajectory coordinates) is conserved if the differential equation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + \dot{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{s}} + \ddot{\boldsymbol{s}} \cdot \frac{\partial}{\partial \dot{\boldsymbol{s}}}\right) Q = 0 \tag{3.1}$$

is satisfied. More generally, a quantity

$$Q = \mathcal{Q}(t, \boldsymbol{s}(t), \dot{\boldsymbol{s}}(t), \ddot{\boldsymbol{s}}(t), \ldots)$$

is conserved if

$$\mathrm{D}_t(Q) = 0 \tag{3.2}$$

where

$$\mathrm{D}_t = \frac{\partial}{\partial t} + \dot{\boldsymbol{s}} \cdot \frac{\partial}{\partial \boldsymbol{s}} + \ddot{\boldsymbol{s}} \cdot \frac{\partial}{\partial \dot{\boldsymbol{s}}} + \cdots. \tag{3.3}$$

For an arbitrary differential equation

$$\Delta(\boldsymbol{x}, u, \partial_{\boldsymbol{x}} u, \ldots) = 0 \tag{3.4}$$

in which we do not distinguish between temporal and spatial independent variables we can generalize further. We consider conserved quantities of the form

$$Q = \mathcal{Q}(\boldsymbol{x}, u, \partial_{\boldsymbol{x}} u, \ldots) \tag{3.5}$$

and our conservation law becomes

$$\sum_{i=0}^{n} \mathrm{D}_i(Q^i) = 0 \tag{3.6a}$$

where

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + s_i^\alpha \frac{\partial}{\partial s^\alpha} + s_{ij}^\alpha \frac{\partial}{\partial s_j^\alpha} + \cdots \tag{3.6b}$$

which must be satisfied for all solutions $u(\boldsymbol{x})$ of (3.4) (Ibragimov 1994, ch 6).

Definition. A conserved quantity, Q, of order n, is a function $\mathcal{Q}(\boldsymbol{x}, u, \partial_{\boldsymbol{x}} u, \ldots, \partial_{\boldsymbol{x}}^n u)$ satisfying (3.6).

In continuum mechanics we often consider MCQs, that is quantities constant on each trajectory; in this case (3.6) is more usually written

$$\left(\frac{\partial}{\partial t} + \dot{\boldsymbol{s}} \cdot \boldsymbol{\nabla}\right) \mathcal{Q}(\boldsymbol{x}, u, \partial_{\boldsymbol{x}} u, \ldots) = 0 \tag{3.7}$$

equality holding on solutions.

One can use equations (3.1)–(3.7) *directly* to find MCQs of a system of equations. In doing this one must remember that as a conservation law, (3.7), must be satisfied for all solutions, $u(x, y, z, t)$, of a differential equation and one must therefore take account of its *frame*.

Definition. The *frame* of a differential equation consists of the equation itself together with its differential consequences.

It may be the case that Q is a function of derivatives of higher order than the highest order terms in the given differential equation, (3.4); we must then take into account not only the equation itself, but also its differential consequences.

3.1. Determining MCQs algorithmically

With the formal definition of a material-conservation law in mind we now describe how one may directly determine MCQs of the system (2.2). There is a considerable amount of algebra involved for third-order and higher laws, so to illustrate the principles involved more clearly we first consider laws of up to second order, i.e. we look for quantities,

$$Q = \mathcal{Q}(\boldsymbol{x}, M, M_{\boldsymbol{x}}, M_{\boldsymbol{xx}}) \tag{3.8}$$

for which (3.7) is satisfied.

Substituting (3.8) into (3.7), using (2.2a, b) and expanding yields

$$\begin{aligned}-M_{yz}\{&Q_x + Q_M M_x + Q_{M_x} M_{xx} + Q_{M_y} M_{xy} + Q_{M_z} M_{xz} + Q_{M_{xx}} M_{xxx}\\
&+Q_{M_{xy}} M_{xxy} + Q_{M_{xz}} M_{xxz} + Q_{M_{yy}} M_{xyy} + Q_{M_{yz}} M_{xyz} + Q_{M_{zz}} M_{xzz}\}\\
&+M_{xz}\{Q_y + Q_M M_y + Q_{M_x} M_{xy} + Q_{M_y} M_{yy} + Q_{M_z} M_{yz} + Q_{M_{xx}} M_{xxy}\\
&+Q_{M_{xy}} M_{xyy} + Q_{M_{xz}} M_{xyz} + Q_{M_{yy}} M_{yyy} + Q_{M_{yz}} M_{yzz} + Q_{M_{zz}} M_{yzz}\}\\
&+\frac{M_x}{y}\{Q_z + Q_M M_z + Q_{M_x} M_{xz} + Q_{M_y} M_{yz} + Q_{M_z} M_{zz} + Q_{M_{xx}} M_{xxz}\\
&+Q_{M_{xy}} M_{xyz} + Q_{M_{xz}} M_{xzz} + Q_{M_{yy}} M_{yyz} + Q_{M_{yz}} M_{yzz} + Q_{M_{zz}} M_{zzz}\} = 0.\end{aligned} \tag{3.9}$$

This equation must hold simultaneously with the frame of (2.2a), however, since in this case we are concerned only with MCQs of up to second order then we do not need to consider differential consequences of (2.2a), only the equation itself. We choose to eliminate M_{zzz} between (2.2a) and (3.9) yielding

$$\begin{aligned}-M_{yz}\{Q_x + Q_M M_x + \cdots + Q_{M_{yz}} M_{xyz}\} + M_{xz}\{Q_y + Q_M M_y + \cdots + Q_{M_{yz}} M_{yzz}\}\\
+\frac{M_x}{y}\{Q_z + Q_M M_z + \cdots + Q_{M_{yz}} M_{yzz}\} = 0.\end{aligned} \tag{3.10}$$

(Care is needed in choosing which term to eliminate since Q and its derivatives potentially depend on some of the terms in (2.2a); Q does not depend on M_{zzz} in this case.) Note the coefficient of $Q_{M_{zz}}$ is identically zero owing to the cancellation of certain terms. Since (3.10) must hold for all solutions $M(x, y, z)$ and Q does not depend on third-order derivatives of M, then the coefficients of products of powers of these third-order derivatives must each be identically zero: from the coefficients of M_{xxx}, M_{xxy}, M_{xxz}, M_{xyy}, M_{xyz}, M_{yyz}, M_{yyy}, M_{xzz} and M_{yzz}, respectively, we obtain

$$M_{yz} Q_{M_{xx}} = 0 \tag{3.11a}$$

$$M_{xz} Q_{M_{xx}} - M_{yz} Q_{M_{xy}} = 0 \tag{3.11b}$$

$$M_x Q_{M_{xx}} - y M_{yz} Q_{M_{xz}} = 0 \tag{3.11c}$$
$$M_{xz} Q_{M_{xy}} - M_{yz} Q_{M_{yy}} = 0 \tag{3.11d}$$
$$y(M_{xz} Q_{M_{xz}} - M_{yz} Q_{M_{yz}}) + M_x Q_{M_{xy}} = 0 \tag{3.11e}$$
$$y M_{xz} Q_{M_{yz}} + M_x Q_{M_{yy}} = 0 \tag{3.11f}$$
$$M_{xz} Q_{M_{yy}} = 0 \tag{3.11g}$$
$$M_x Q_{M_{xz}} = 0 \tag{3.11h}$$
$$M_x Q_{M_{yz}} = 0. \tag{3.11i}$$

In general M_x, M_{xz} and M_{yz} are nonzero, so from (3.11) we conclude that

$$Q_{M_{xx}} = Q_{M_{xy}} = Q_{M_{xz}} = Q_{M_{yy}} = Q_{M_{yz}} = 0. \tag{3.12}$$

Note that we have *not* shown that $Q_{M_{zz}} = 0$ (recall the cancellation of terms mentioned above); indeed it turns out that $Q_{M_{zz}} \neq 0$.

Using (3.12) we find that (3.10) simplifies substantially:

$$-M_{yz}\{Q_x + Q_M M_x + Q_{M_x} M_{xx} + Q_{M_y} M_{xy}\} + M_{xz}\{Q_y + Q_M M_x + Q_{M_x} M_{xx} + Q_{M_y} M_{yy}\} + \frac{M_x}{y}\{Q_z + Q_M M_z + Q_{M_x} M_{xz} + Q_{M_y} M_{yz} + Q_{M_z} M_{zz}\} = 0. \tag{3.13}$$

Since we have determined that Q depends on only one second-order derivative of M, M_{zz}, then the coefficients of products of powers of all other second-order derivatives must each be identically zero. From the coefficients of $M_{xx}M_{yz}$, $M_{yy}M_{xz}$, $M_{yz}M_{xy}$ and $M_{xz}M_{xy}$ we see that

$$Q_{M_x} = Q_{M_y} = 0 \tag{3.14}$$

and then using this result, from the coefficients of M_{xz} and M_{yz} we find

$$Q_y + Q_M M_y = 0 \tag{3.15a}$$
$$Q_x + Q_M M_x = 0 \tag{3.15b}$$
$$\frac{M_x}{y}\{Q_z + Q_M M_z + Q_{M_z} M_{zz}\} = 0. \tag{3.15c}$$

From (3.14) we know that Q is independent of M_x and M_y, so that from (3.15*a*) and (3.15*b*) we find

$$Q_x = Q_y = Q_M = 0. \tag{3.16}$$

It remains to satisfy (3.15*c*). M_x is not in general zero and then since $Q_M = 0$ then we require the solution of $Q_z + Q_{M_z} M_{zz} = 0$; there are two cases to consider: (i), $Q_z = Q_{M_z} = 0$ and (ii), $Q_z Q_{M_z} \neq 0$. In case (i) the general solution is $Q = \mathcal{Q}(M_{zz})$ and in case (ii) it is $Q = \mathcal{Q}(M_z - zM_{zz})$. Hence all second-order, MCQs of system (2.2) are of the form

$$Q = \mathcal{Q}(M_{zz}, M_z - zM_{zz}). \tag{3.17}$$

It is easy to check that all quantities of this form are indeed materially conserved.

We now turn to the problem of determining all MCQs of up to third order, i.e. we assume

$$Q = \mathcal{Q}(x, y, z, M, M_x, M_y, M_z, M_{xx}, M_{xy}, \ldots, M_{zz}, M_{xxx}, M_{xxy}, \ldots, M_{zzz}). \tag{3.18}$$

Substituting this into (3.7) we obtain

$$-M_{yz}\{Q_x + Q_M M_x \cdots + Q_{M_{zzz}} M_{xzzz}\} + M_{xz}\{Q_y + Q_M M_y \cdots + Q_{M_{zzz}} M_{yzzz}\} + \frac{M_x}{y}\{Q_z + Q_M M_z \cdots + Q_{M_{zzz}} M_{zzzz}\} = 0 \tag{3.19}$$

and this must hold simultaneously with the frame of (2.2*a*) (cf (3.9), above). This time, since (3.19) contains derivatives of M of up to fourth order we must consider not only (2.2*a*) itself, but in addition its derivative with respect to each independent variable, x, y and z:

$$M_{xx}M_{zzz} + M_x M_{xzzz} + y(M_{xxz}M_{yzz} + \cdots - M_{yz}M_{xxzz}) = 0 \tag{3.20a}$$

$$M_{xy}M_{zzz} + M_x M_{yzzz} + M_{xz}M_{yzz} - M_{yz}M_{xzz} + y(M_{xyz}M_{yzz} + \cdots - M_{yz}M_{xyzz}) = 0 \tag{3.20b}$$

$$M_{xz}M_{zzz} + M_x M_{zzzz} + y(M_{xzz}M_{yzz} + \cdots - M_{yz}M_{xzzz}) = 0 \tag{3.20c}$$

respectively. The procedure is to use (2.2*a*), (3.20*a*–*c*) to eliminate four terms from (3.19) and then continue in a similar way to the second-order case above. Again we must proceed carefully as the derivatives of Q in (3.19) potentially depend upon all derivatives of M up to third order and we cannot eliminate implicit terms! We first choose to eliminate M_{xxzz}, M_{yyzz} and M_{zzzz} from (3.19) by using (3.20*a*–*c*), respectively; we may do this without difficulty, Q being independent of fourth-order derivatives of M. We obtain

$$\begin{aligned} Q_{M_{xzz}}\{M_{xx}M_{zzz} + M_x M_{xzzz} + y(M_{xxz}M_{yzz} + M_{xz}M_{xyzz} - M_{xyz}M_{xzz})\} \\ +Q_{M_{yzz}}\{M_{yz}M_{xzz}\cdots + y(M_{yyz}M_{xzz} + \cdots)\} \\ -M_{yz}\{Q_x + Q_M M_x + Q_{M_x}M_{xx} + \cdots + Q_{M_{zzz}}M_{xzzz}\} \\ +M_{xz}\{Q_y + Q_M M_y + Q_{M_x}M_{xy} + \cdots + Q_{M_{zzz}}M_{yzzz}\} \\ +\frac{M_x}{y}\left\{\frac{Q_{M_{xxz}}}{M_{yz}}[M_{xx}M_{zzz} + \cdots + y(M_{xxz}M_{yzz} + \cdots)]\right. \\ +\frac{Q_{M_{yyz}}}{M_{xz}}[M_{yz}M_{xzz} - \cdots + y(M_{yyz}M_{xzz} + \cdots)] \\ +\frac{Q_{M_{zzz}}}{M_x}[y(M_{yz}M_{xzzz} - M_{xz}M_{yzzz}) - M_{xz}M_{zzz}] \\ \left. +Q_z + Q_M M_z + \cdots + Q_{M_{yzz}}M_{yzzz}\right\} = 0. \end{aligned} \tag{3.21}$$

Note that owing to the cancellation of certain terms the coefficient of $Q_{M_{zzz}}$ is zero. Since (3.21) must be satisfied for all solutions $M(x, y, z)$ of (2.2*a*) then the coefficients of products of powers of each fourth-order derivative must be identically zero (in drawing conclusions from these identities we must take account of (2.2*a*), at least in principle, as we have considered only (3.20*a*–*c*) so far). From the coefficients of M_{xxxx}, M_{yyyy}, M_{xxxy}, M_{xxxz}, M_{xxyy}, M_{xxyz}, M_{xyyy}, M_{xyyz}, M_{xzzz}, M_{yzzz}, M_{yyyz} and M_{xyzz} we obtain respectively

$$-M_{yz}Q_{M_{xxx}} = 0 \tag{3.22a}$$

$$-M_{xz}Q_{M_{yyy}} = 0 \tag{3.22b}$$

$$-M_{yz}Q_{M_{xxy}} + M_{xz}Q_{M_{xxx}} = 0 \tag{3.22c}$$

$$-M_{yz}Q_{M_{xxz}} = 0 \tag{3.22d}$$

$$-M_{yz}Q_{M_{xyy}} + M_{xz}Q_{M_{xxy}} = 0 \tag{3.22e}$$

$$-M_{yz}Q_{M_{xyz}} + M_{xz}Q_{M_{xxz}} + \frac{M_x}{y}Q_{M_{xxy}} = 0 \tag{3.22f}$$

$$-M_{yz}Q_{M_{yyy}} + M_{xz}Q_{M_{xyy}} = 0 \tag{3.22g}$$

$$-M_{yz}Q_{M_{yyz}} + M_{xz}Q_{M_{xyz}} + \frac{M_x}{y}Q_{M_{xyy}} = 0 \tag{3.22h}$$

$$-\left(1-\frac{1}{y}\right) M_x Q_{M_{xzz}} + \frac{M_x^2 Q_{M_{xxz}}}{y M_{yz}} = 0 \tag{3.22i}$$

$$-M_x Q_{M_{yzz}} - \frac{M_x^2 Q_{M_{yyz}}}{y M_{xz}} + \frac{M_x}{y} Q_{M_{yzz}} = 0 \tag{3.22j}$$

$$M_{xz} Q_{M_{yyz}} + \frac{M_x}{y} Q_{M_{yyy}} = 0 \tag{3.22k}$$

$$-y M_{xz} Q_{M_{xzz}} - M_{yz} Q_{M_{yzz}} + y M_{yz} Q_{M_{yzz}} + M_{xz} Q_{M_{xzz}} + \left(\frac{y M_{xz} Q_{M_{xxz}}}{M_{yz}} + \frac{y M_{yz} Q_{M_{yyz}}}{M_{xz}} + Q_{M_{xyz}}\right) \frac{M_x}{y} = 0. \tag{3.22l}$$

In general M_x, M_{xz} and M_{yz} are nonzero, so from (3.22) we conclude that

$$Q_{M_{xxx}} = Q_{M_{yyy}} = Q_{M_{xxy}} = Q_{M_{xxz}} = Q_{M_{xyy}} = Q_{M_{xyz}} = Q_{M_{yyz}} = Q_{M_{xzz}} = Q_{M_{yzz}} = 0. \tag{3.23}$$

Note that we have *not* shown that $Q_{M_{zzz}} = 0$ as no coefficient of a fourth-order derivative of M in (3.21) involves $Q_{M_{zzz}}$ (recall the cancellation of terms mentioned above); indeed it turns out that in general $Q_{M_{zzz}} \neq 0$. In the special case $Q_{M_{zzz}} = 0$ we reduce to the search for second-order MCQs considered above. We proceed with the general case.

It turned out that in obtaining (3.23) the condition (2.2*a*) is not needed; since we are now sure that Q is independent of both M_{xzz} and M_{yzz} we can use (2.2*a*) to eliminate one of these third-order terms from (3.21). Choosing M_{xzz} and using (3.23) then (3.21) becomes

$$\begin{aligned}
-M_{yz}\{&Q_x + Q_M M_x + Q_{M_x} M_{xx} + Q_{M_y} M_{xy} + Q_{M_z} M_{xz} + Q_{M_{xx}} M_{xxz} + Q_{M_{xy}} M_{xxy} \\
&+ Q_{M_{xz}} M_{xzz} + Q_{M_{yy}} M_{xyy} + Q_{M_{yz}} M_{xyz} + Q_{M_{zz}} M_{xzz} \\
&+ \frac{Q_{M_{zz}}}{y M_{yz}} [M_x M_{zzz} + y M_{xz} M_{yzz}]\} \\
&+ M_{xz}\{Q_y + Q_M M_y + Q_{M_x} M_{xy} + Q_{M_y} M_{yy} + Q_{M_z} M_{yz} + Q_{M_{xx}} M_{xxy} \\
&+ Q_{M_{xy}} M_{xyy} + Q_{M_{xz}} M_{xyz} + Q_{M_{yy}} M_{yyy} + Q_{M_{yz}} M_{yzz} + Q_{M_{zz}} M_{yzz}\} \\
&+ \frac{M_x}{y}\{Q_z + Q_M M_z + Q_{M_x} M_{xz} + Q_{M_y} M_{yz} + Q_{M_z} M_{zz} + Q_{M_{xx}} M_{xxz} \\
&+ Q_{M_{xy}} M_{xyz} + Q_{M_{xz}} M_{xzz} + Q_{M_{yy}} M_{yyz} + Q_{M_{yz}} M_{yzz} + Q_{M_{zz}} M_{zzz}\} \\
&- \frac{1}{y} M_{xz} M_{zzz} Q_{M_{zzz}} = 0.
\end{aligned} \tag{3.24}$$

(We find that there are no terms involving $Q_{M_{zz}}$ owing to cancellation of terms.) Since we have shown that Q is independent of all but one third-order derivative of M then the coefficients of different products of powers of third-order terms in (3.24), except those involving M_{zzz} must be identically zero:

$$-M_{yz} Q_{M_{xx}} = 0 \tag{3.25a}$$

$$-M_{yz} Q_{M_{xy}} + M_{xz} Q_{M_{xx}} = 0 \tag{3.25b}$$

$$-M_{yz} Q_{M_{xz}} + \frac{M_x}{y} Q_{M_{xx}} = 0 \tag{3.25c}$$

$$-M_{yz} Q_{M_{yy}} + M_{xz} Q_{M_{xy}} = 0 \tag{3.25d}$$

$$-M_{yz} Q_{M_{yz}} + M_{xz} Q_{M_{xz}} + \frac{M_x}{y} Q_{M_{yz}} = 0 \tag{3.25e}$$

$$\frac{M_x}{y} Q_{M_{yz}} = 0 \tag{3.25f}$$

$$M_{xz}Q_{M_{yy}} = 0 \tag{3.25g}$$

$$M_{xy}Q_{M_{yz}} + \frac{M_x}{y}Q_{M_{yy}} = 0 \tag{3.25h}$$

$$\frac{M_x}{y}Q_{M_{xz}} = 0. \tag{3.25i}$$

In general M_x, M_{xz} and M_{yz} are nonzero, so from (3.25) we conclude that

$$Q_{M_{xx}} = Q_{M_{xy}} = Q_{M_{xz}} = Q_{M_{yy}} = Q_{M_{yz}} = 0. \tag{3.26}$$

Now, since $Q_{M_{xx}} = Q_{M_{yy}} = 0$ then the coefficients of different products of powers of the corresponding second-order derivatives of M in (3.25) must be identically zero and so, from the coefficients of M_{xx} and M_{yy}, we obtain

$$-M_{yz}Q_{M_x} = 0 \tag{3.27}$$

$$M_{xz}Q_{M_y} = 0 \tag{3.28}$$

respectively, from which we conclude

$$Q_{M_x} = Q_{M_y} = 0. \tag{3.29}$$

Using results (3.26) and (3.29) then (3.24) simplifies considerably:

$$-M_{yz}(Q_x + Q_M M_x) + M_{xz}\left\{Q_y + Q_M M_y - \frac{1}{y}M_{zzz}Q_{M_{zzz}}\right\} - \frac{M_x}{y}\{Q_z + Q_M M_z + Q_{M_z}M_{zz}\} = 0. \tag{3.30}$$

Finally, since $Q_{M_{yz}} = Q_{M_x} = 0$ then from the coefficients of $M_x M_{yz}$ and M_{yz} we obtain

$$Q_M = Q_x = 0 \tag{3.31}$$

leaving

$$M_{xz}\left\{Q_y - \frac{1}{y}M_{zzz}Q_{M_{zzz}}\right\} + \frac{M_x}{y}\{Q_z + M_{zz}Q_{M_z}\} = 0. \tag{3.32}$$

Since in general $M_x M_{xz} \neq 0$ then

$$Q_y - \frac{1}{y}M_{zzz}Q_{M_{zzz}} = 0 \tag{3.33a}$$

$$Q_z + M_{zz}Q_{M_z} = 0. \tag{3.33b}$$

The general solutions to these equations are respectively $Q = \mathcal{Q}_a(yM_{zzz}, c_a)$, $Q = \mathcal{Q}_b(M_z - zM_{zz}, M_{zz}, c_b)$, where c_a and c_b are constants of integration; therefore the general MCQ of the system (2.2) is

$$Q = \mathcal{Q}(yM_{zzz}, M_z - zM_{zz}, M_{zz}). \tag{3.34}$$

It is easy to check that all quantities of this form are indeed materially conserved.

3.2. The question of further MCQs

Using an essentially algorithmic method we have found three MCQs of the system (2.2) or order 3 or less. Are there other such quantities, of higher order? Using the method, what happens when there are no MCQs corresponding to the ansatz being studied?

Recall our definition of a MCQ, (3.7). In the context of (2.2) this becomes

$$\left\{-\frac{M_{yz}}{y}\frac{\partial}{\partial x} + \frac{M_{xz}}{y}\frac{\partial}{\partial y} + \frac{M_x}{y^2}\frac{\partial}{\partial z}\right\}Q = 0 \tag{3.35}$$

which may be written (informally) as

$$\frac{\mathrm{d}x}{-M_{yz}/y} = \frac{\mathrm{d}y}{M_{xz}/y} = \frac{\mathrm{d}z}{M_x/y} = \frac{\mathrm{d}Q}{0}. \tag{3.36}$$

The general solution of (3.36) includes exactly three 'constants' of integration so that the general solution of (3.35) is necessarily of the form

$$Q = \mathcal{S}(C_1, C_2, C_3) \tag{3.37}$$

where C_1, C_2 and C_3 are functions of y and derivatives of M, i.e. Q can at most be a function of three arguments. Hence there are no MCQs of fourth order or higher and (3.34) is indeed the most general one (we discuss this idea for a general fluid-dynamical system in section 5).

It is worth looking briefly at how the method would show that there were no (further) MCQs corresponding to a particular ansatz, for example, for (2.2), that there are no fourth-order quantities (one will not always be as fortunate as for the system above!). Rather than consider ansatzes of the form $Q = \mathcal{Q}(\boldsymbol{x}, M, M_{\boldsymbol{x}}, M_{\boldsymbol{xx}}, M_{\boldsymbol{xxx}}, M_{\boldsymbol{xxxx}})$, or considering an alternative system of governing equations, we can see what will happen by looking again at the computations of section 3.1. When looking for MCQs of up to second order we noted that the coefficient of $Q_{M_{zz}}$ was identically zero (see text between equations (3.10) and (3.11)); also, when looking for MCQs of up to third order we noted that the coefficient of $Q_{M_{zzz}}$ was identically zero (see text between equations (3.21) and (3.22)). These fortuitous events lead to (3.15*c*), and to (3.33*a*) and (3.33*b*), the final systems of equations for MCQs, Q. Were this *not* the case then one would have obtained equations forcing the conclusion that $Q_{M_{zz}} = Q_{M_{zzz}} = 0$ (cf equations (3.11) and (3.22)), and our final equations would be, respectively,

$$\frac{M_x}{y}\{Q_z + Q_M M_z\} = 0$$

and

$$Q_y = Q_z = 0$$

from which we must conclude that there are no MCQs. In short, the fortuitous cancellation of certain terms from the computation leading to 'zero coefficients' for corresponding derivatives of Q, as one proceeds, leads to the existence of MCQs—should such 'zero coefficients' not appear then nor will MCQs.

4. Constructing solutions

In this section we consider the construction of solutions of the governing equations, (2.2), from the functionally related MCQs, i.e. from (2.4). First we consider an example solution, in an ocean basin, which is of particular physical significance and illustrates the generality of solutions obtainable from (2.4). The example shows that solutions constructed by means of the MCQs satisfy *all* the boundary conditions which are usually applied to the problem. Previous analytical solutions are either unable to satisfy a full set of general boundary conditions (e.g. Salmon and Hollerbach 1991, Hood 1996, Hood and Williams 1996) or impose a particular physical character on the solution (e.g. Robinson and Stommel 1959, Welander 1959, Needler 1967, 1971, Hodnett 1978); either way generality is lost. Second we consider how one might systematically determine the function relating the MCQs, $\mathcal{M}$ in (2.4), by generalizing (2.2) in a physically motivated way.

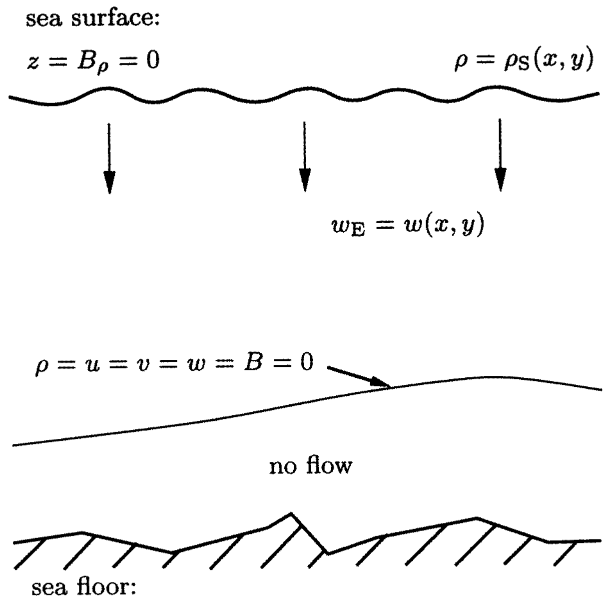

Figure 1. A schematic west–east vertical section through the northern Atlantic Ocean showing the boundary conditions usually applied to the model: at the sea-surface, $z = B_\rho = 0$, the vertical velocity, $w_E = w(x, y)$ and, optionally, the surface density distribution, $\rho_S = \rho(x, y)$, are imposed; at depth, a level of no-motion may be assumed; the western boundary is passive, but at the eastern boundary an integral condition is applied (see text); in addition the density profile may be prescribed at certain locations, (x, y).

4.1. An example—an ocean basin

We now describe how, using solutions determined from (2.4), one can determine the ocean dynamics and thermodynamics in the northern Atlantic Ocean (for a full description of the problem see Hood and Williams (1996) and references contained therein). Figure 1 shows a schematic west–east vertical section through the region of interest with the boundary conditions which are to be applied. It is not clear, either from a physical or mathematical standpoint, what the function $\mathcal{M}$ should be. However, previous work on special cases suggests we take the following approach.

First, following Killworth (1987), we change to a coordinate system in which we use density, ρ, as the 'vertical' coordinate, rather than depth, z. Then, choosing $B(x, y, \rho) = p + \rho z$, the Bernoulli function, as the dependent variable in place of $M(x, y, z)$, the governing equations, (2.2), become

$$u = -\frac{B_y}{y} \qquad v = \frac{B_x}{y} \qquad z = B_\rho \qquad w = u z_x + v z_y \tag{4.1a}$$

$$(B_x B_{y\rho\rho} - B_y B_{x\rho\rho})y - B_x B_{\rho\rho} = 0 \tag{4.1b}$$

the potential vorticity is

$$q = \frac{y}{B_{\rho\rho}} \tag{4.2}$$

and so (2.4) becomes

$$B_{\rho\rho} = y\mathcal{B}(B, \rho) \tag{4.3}$$

(a considerable simplification).

The simplest function $\mathcal{B}$ is linear giving

$$B_{\rho\rho} = f_1(\rho)B + f_0(\rho) \tag{4.4}$$

where both $f_1(\rho)$ and $f_0(\rho)$ are to be determined. In fact this simple case is of particular physical significance: Killworth (1987) considered the case $f_0 = 0$ with the restricted boundary conditions $\rho = \rho_S(y)$ (i.e. the surface density is a function of only latitude) with $w = w_E(y)$ (the surface vertical velocity is also a function of latitude, only) and/or $u(0, y, z) = 0$ (eastern velocity at the eastern boundary is zero at all depths and latitudes); Salmon (1994) considered $f_1 = 0$ and added time dependence to the problem.

From (4.4) we expect to find two functions of integration, $a(x, y)$ and $b(x, y)$; we also have to fix $f_1(\rho)$ and $f_0(\rho)$. The boundary conditions for this problem are a subject of some debate; we therefore consider two cases. Equations (2.2) are a hyperbolic system (Huang 1988) so one cannot expect to apply conditions at all boundaries: the northern, southern and western boundaries are all passive. We apply conditions at the surface, bottom and an integral condition at the eastern boundary. (For more detailed discussion of boundary conditions for this problem see Killworth (1987), Samelson and Vallis (1997), Huang (1988).)

Case (i). We impose the vertical velocity field at the surface (more accurately, below some surface layer, subject to different dominant dynamics, which we patch onto our solution domain), i.e. we prescribe $w_E(x, y)$, where

$$w_E(x, y) = \frac{1}{y}(B_x B_{y\rho} - B_y B_{x\rho})|_{B_\rho=0}. \tag{4.5a}$$

We also assume that flow below some depth, which we take to be $\rho = 0$, without loss of generality, is an order of magnitude weaker than in our solution domain; this is described by

$$B(x, y, 0) = 0 \tag{4.5b}$$

(Killworth 1987, section 2).

Case (ii). Alternatively, we can prescribe both w and ρ at the surface, i.e.

$$w_E(x, y) = \frac{1}{y}(B_x B_{y\rho} - B_y B_{x\rho})|_{B_\rho=0} \tag{4.6a}$$

$$\rho_S(x, y) = \rho|_{B_\rho=0}. \tag{4.6b}$$

In fact these conditions are not enough to fix $a(x, y)$ and $b(x, y)$. In each case we have imposed one algebraic constraint, so that a and b are no longer independent, together with one first-order PDE with independent variables x and y. Consequently it remains to fix an arbitrary function of some known function of x and y resulting from the integration of the first-order PDE. This can be done by placing an (integral) constraint on the flow or heat flux through the eastern boundary. Finally, $f_1(\rho)$ and $f_0(\rho)$ may be set by prescribing the variation of density with depth at two points (x, y).

4.2. Systematically finding $\mathcal{M}$

Above we considered a particular case of $\mathcal{M}$ in (2.4) (i.e. $\mathcal{B}$ in (4.3)), motivated by other authors' work. The question remains, how given a particular problem, would one find $\mathcal{M}$ systematically? An answer, it turns out, is to make the model which the governing equations (2.2) represents *more* realistic.

Equations (2.2) represent an ideal system, i.e. a system in which diffusion, friction and other such effects are assumed negligible. This is done for simplicity and is only approximately true. What happens if we introduce some nonideal effect, parametrized by κ, $0 < \kappa \ll 1$? The relation (4.3) will now be only approximately true. We suppose

$$B_{\rho\rho} = y\mathcal{B}(\rho, B, \kappa x, \kappa y) \tag{4.7}$$

i.e. the nonideal effect is balanced, mathematically, by a slow variation in the functional relation between the MCQs.

For illustration let us consider Fickian diffusion, so that, (4.1*b*) becomes

$$(B_x B_{y\rho\rho} - B_y B_{x\rho\rho})y - B_x B_{\rho\rho} = \kappa y^2 B_{\rho\rho\rho} \tag{4.8}$$

and let us work with the special case studied by Salmon (1994), i.e.

$$B_{\rho\rho} = y\mathcal{F}(\rho). \tag{4.9}$$

Generalizing (4.9) to include slow variation with x and y, and integrating we obtain

$$B(x, y, \rho) = yF(\rho, \kappa x, \kappa y) + \rho b(x, y) + a(x, y) \qquad F(\rho) = \int^{\rho}\int^{\rho_1} \frac{1}{\mathcal{Q}(\rho_2)}\, \mathrm{d}\rho_2\, \mathrm{d}\rho_1 \tag{4.10}$$

(where a and b are functions of integration, to be determined), then substituting into (4.8) terms of O(1) cancel leaving

$$(\rho b_x + a_x)F_{Y\rho\rho} = y^2 F_{\rho\rho\rho} + \mathrm{O}(\kappa^2) \qquad Y = \kappa y. \tag{4.11}$$

This differential constraint does not fully determine F, or equivalently $\mathcal{F}$, but perhaps points the way. This approach, for both Salmon's special case, (4.9), and the general case is actively being considered by the author (Hood 1997).

5. Discussion and conclusions

In the preceding sections we have considered how one might use the conservation laws of the governing PDEs describing a system to construct exact analytical solutions; we have focused on the equations describing large-scale flow within the ocean (the geostrophic equations). For these equations we used a *systematic* method to recover three MCQs which were necessarily functionally related; this relation takes the form of a differential equation which is significantly simpler than the governing equations, the solution of which can be used as a quite general solution ansatz for the governing PDEs. We also proved that no more MCQs exist. Finally we considered an example, an ocean basin, which illustrated the generality of the solutions obtained and suggested a mechanism reflecting higher-order dynamics by which one might determine the function relating the quantities.

Generality of solutions. It is important to emphasize that solutions obtained by means of relating MCQs are significantly more general than those obtained by other analytical methods, for these equations. In early work on (2.2) informed guesses where made for forms of solution sometimes based on similarity variables or partial separation of variables, e.g. Robinson and Stommel (1959), Welander (1959), Needler (1967, 1971) Hodnett (1978). These works mark significant analytical progress on a formidable system of equations. However, in each case there are limitations on boundary conditions which can be satisfied and the physical character of the solutions, perhaps motivated by the physics, is built into the solution method. An alternative, systematic (indeed algorithmic) method has been tried: Salmon and Hollerbach (1991) used Lie's method (e.g. Bluman and Kumei 1989, Olver 1986, Stephani 1990) to determine a set of point symmetries of (2.2), with Fickian diffusion added in (2.2c) (cf (4.8)). This work was extended, for more general diffusion, using a more general method which obtained more symmetries by Hood (1996) and Hood and Williams (1996). Further, the time-dependent problem was studied (using Lie's method) by Edwards (1996). These symmetries were used to construct classes of solutions which were general enough to carry out simple experiments to investigate particular physical processes, however, again, only a limited class of boundary conditions can be satisfied. In contrast, integration of (2.4) (equivalently (4.3)) yields the general solution of the governing equations, (2.2) (equivalently (4.3)), in the sense that there are two arbitrary functions of x and y and (at least) one of z (or ρ). As illustrated in section 4.1 these solutions can satisfy the full set of boundary conditions that one would expect to impose.

Wider application. These ideas are certainly not restricted to the geostrophic equations. In general, given a system of PDEs in n independent variables with n MCQs known, one can construct a solution ansatz in exactly the same way. For example consider the 2D Euler equations,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{g} - \frac{1}{\rho}\boldsymbol{\nabla} p. \tag{5.1}$$

Both vorticity, ω (where $\omega\boldsymbol{k} = \boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$), and the stream function, ψ (where $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, which automatically ensures mass conservation), are materially conserved, whence

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \mathcal{G}(\psi) \tag{5.2}$$

for some function $\mathcal{G}$. Equation (5.2) alone determines the flow, given $\mathcal{G}$. Certain functions have been considered (e.g. many workers have assumed a linear q–ψ relationship, but there is analytical, numerical and experimental evidence that this is not always possible, especially in the case of a 'tripolar' vortex (van Heijst *et al* 1991, Legras *et al* 1988)); as in section 4.2 one could add a small amount of some nonideal effect to suggest what $\mathcal{G}$ might be.

It is likely that for any system of equations for which these ideas are fruitfully employed, the resulting analytical solutions will be significantly more general than those resulting from point symmetries—stronger boundary conditions will be satisfied. Furthermore, one may be able to construct useful solutions even with fewer than n MCQs known—indeed, the most physically relevant case may involve fewer than n (e.g. Salmon 1994).

In section 3.2 we showed that the geostrophic equations, (2.2), have no more than three MCQs. This principle carries over to any fluid-dynamical system. An MCQ must satisfy

$$\left\{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right\} Q = 0 \tag{5.3}$$

(for a steady system), which may be written (informally) as

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} = \frac{\mathrm{d}Q}{0}. \tag{5.4}$$

In principle, one can always solve the governing equations for $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$, and then integrate (5.4) for Q as a function of three 'constants' of motion. This is true for any $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$. It follows that a three-dimensional, steady system has three MCQs. This principle is readily extended to systems with more (or fewer) independent variables. However, to be of interest, a quantity must be materially conserved for a wide class of solutions of the governing equations (preferably all solutions), i.e. we want to be able to find MCQs without first solving the governing equations—indeed, we may well intend that knowledge of the MCQs is to help us solve them! If the equations describing a fluid-dynamical system can be put in potential form, cf (2.2), then we have a much better chance of finding *useful* MCQs. In this case we can construct an equation such as (3.35). It may be possible to integrate this directly, though this is unlikely. Usually finding the MCQs by using the method described in section 3.1 will be the most fruitful approach: given that the number of MCQs is known from the number of independent variables then one can consider candidates of increasing order, using a computer algebra system, until all these are found.

Relation of point-symmetry solutions and MCQ solutions; Noether. The question arises, what is the connection between solutions obtained by means of point symmetries (of the governing equations) and those constructed from MCQs? Is one a subset of the other? Noether's Theorem (e.g. Bluman and Kumei 1989, Stephani 1990) gives us some insight.

Solutions obtainable from material-conservation laws are a consequence of the existence of point symmetries of *the action integral* of a system; in contrast, the usual point-symmetry methods determine solutions which reflect symmetries of the governing equations themselves. Since the solutions constructed from MCQs are general in the sense stated above then one would usually expect solutions constructed from point symmetries to be special cases of the former—of course some of the point-symmetry solutions may well be singular.

Noether's theorem suggests an alternative approach to the construction of conservation laws to that advocated in this paper: determine the point symmetries of the action integral and use these to construct the MCQs. This approach is also essentially algorithmic, however, one is limited to those systems for which a Lagrangian is known. This is a significant weakness from which the more direct approach of section 3 does not suffer.

Backwards. Finally, one can turn the whole problem around. Given that one can determine the MCQs of a fluid-dynamical system from the velocity field, an obvious question that arises is, given n MCQs of a system in n independent variables, can one determine the corresponding dynamics? Yes—but not uniquely. Let us return to the three MCQs given by (2.3). Any velocity field for which these three quantities are materially conserved must satisfy

$$uM_{xzz} + vM_{yzz} + wM_{zzz} = 0 \tag{5.5a}$$

$$u(M_{xz} - zM_{xzz}) + v(M_{yz} - zM_{yzz}) - wzM_{zzz} = 0 \tag{5.5b}$$

$$uyM_{xzzz} + v(M_{zzz} + yM_{yzzz}) + wyM_{zzzz} = 0 \tag{5.5c}$$

which is a 3×3 linear, algebraic system for u, v and w. Following some straightforward algebra we find a solution exists provided

$$M_{xz}M_{zzz}^2 + yM_{zzzz}(M_{yz}M_{xzz} - M_{xz}M_{yzz}) + yM_{zzz}(M_{xz}M_{yzzz} - M_{yz}M_{xzzz}) = 0 \tag{5.6}$$

and then the general solution, for u, v and w may be written,

$$M_{xz}u + M_{yz}v = 0 \tag{5.7a}$$

$$w = -\frac{M_{xzz}u + M_{yzz}v}{M_{zzz}}. \tag{5.7b}$$

An integrating factor for (5.6) is easily found: multiplying through by M_{zzz}^{-2} and integrating we find

$$M_x M_{zzz} + y(M_{xz}M_{yzz} - M_{yz}M_{xzz}) = M_0(x, y)M_{zzz} \tag{5.8}$$

where $M_0(x, y)$ is a function of integration. The dynamics corresponding to the MCQs (2.3) is therefore given by (5.7) and (5.8). Equation (5.8 is in fact equivalent to (2.2a) (substitute $M(x, y) + \int^x M_0(\hat{x}, y)\,\mathrm{d}\hat{x}$ for $M(x, y)$); this equivalence reflects a point symmetry of (2.2a) (Salmon and Hollerbach 1991). Nevertheless, a degree of freedom in the velocity field remains—the dynamics are not uniquely determined by the MCQs.

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